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LETTER TO THE EDITOR

Continuous Hahn polynomials

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Abstract. A slightly more general orthogonality relation for the Hahn polynomials of a continuous variable than the recent one given by Atakishiyev and Suslov, is given here.

Hahn polynomials are one set of orthogonal polynomials that contain a transformed form of $3j$-symbols and one of the orthogonality relations of the $3j$-symbols. They are usually given as

$$Q_n(x;\alpha,\beta,N)={}_3F_2\left[\begin{matrix}-n,\ n+\alpha+\beta+1,\ -x\\ \alpha+1,\ -N\end{matrix};1\right] \tag{1}$$

and the orthogonality relation is

$$\sum_{x=0}^{N} Q_n(x;\alpha,\beta,N)Q_m(x;\alpha,\beta,N)\begin{bmatrix}x+\alpha\\ x\end{bmatrix}\begin{bmatrix}N-x+\beta\\ N-x\end{bmatrix}=0, \qquad m\neq n\leqslant N. \tag{2}$$

The reader should be made aware that the notation differs depending on the author (or even the paper, since some authors have changed notation). In particular, the N above is $N-1$ in Atakishiyev and Suslov (1985). These authors considered the above polynomials as functions of a complex variable, and for appropriate complex N, found an orthogonality relation whose measure is absolutely continuous. There is a more general complex measure which is given below.

Set

$$\begin{aligned}P_n(x)&=p_n(x;a,b,c,d)\\ &=\mathrm{i}^n\frac{(a+c)_n(a+d)_n}{n!}\,{}_3F_2\left[\begin{matrix}-n,\ n+a+b+c+d-1,\ a-\mathrm{i}x\\ a+c,\ a+d\end{matrix};1\right]\end{aligned} \tag{3}$$

where $(a)_n=\Gamma(n+a)/\Gamma(a)$ is the usual shifted factorial. Then

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)w(x)\,\mathrm{d}x=\frac{\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)}{(2n+a+b+c+d-1)\Gamma(n+a+b+c+d-1)}\,\delta_{m,n}, \tag{4}$$

where the weight function $w(x)$ is given by

$$2\pi w(x)=\Gamma(a+\mathrm{i}x)\Gamma(b+\mathrm{i}x)\Gamma(c-\mathrm{i}x)\Gamma(d-\mathrm{i}x), \tag{5}$$

and the contour is the real axis if the real part of each parameter is positive, and is deformed to separate the increasing sequences of poles from the decreasing sequences of poles in the general case. This can be done when $(a+c)$, $(a+d)$, $(b+c)$ and $(b+d)$

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are not zero or negative integers. The integrals below will be written as $\int_{-\infty}^{\infty}$ but should be interpreted as above if necessary.

To prove (4), it is sufficient to show that $P_n(x)$ is orthogonal to one polynomial of each degree less than n, and to check the constant when $m=n$. To show the first, use Barnes's beta integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(a+\mathrm{i}x)\Gamma(b+\mathrm{i}x)\Gamma(c-\mathrm{i}x)\Gamma(d-\mathrm{i}x)\mathrm{d}x=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

(see Bailey 1964, pp 6 and 7) to obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} & P_n(x)(b+\mathrm{i}x)_m w(x)\mathrm{d}x\\ &=\frac{\mathrm{i}^n(a+c)_n(a+d)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+a+b+c+d-1)_k}{(a+c)_k(a+d)_k k!}\\ &\quad\times\frac{\Gamma(k+a+c)\Gamma(k+a+d)\Gamma(m+b+c)\Gamma(m+b+d)}{\Gamma(k+m+a+b+c+d)}\\ &=\frac{\mathrm{i}^n\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(m+b+c)\Gamma(m+b+d)}{n!\Gamma(m+a+b+c+d)}\\ &\quad\times{}_2F_1\left[\begin{matrix}-n,\ n+a+b+c+d-1\\ m+a+b+c+d\end{matrix};1\right].\end{aligned}$$

But

$${}_2F_1\left[\begin{matrix}-n,\ n+a+b+c+d-1\\ m+a+b+c+d\end{matrix};1\right]=\frac{(1+m-n)}{(m+a+b+c+d)_n}$$

and $(1+m-n)_n=(-1)^n(-m)_n$ (Bailey 1964, p3), so this integral vanishes when $m=0,1,\ldots,n-1$. The constant when $m=n$ comes from the observation that

$$(b+\mathrm{i}x)_n=(a+\mathrm{i}x)_n+(\text{polynomial of degree } n-1),$$

the orthogonality just proved and the definition of $p_n(x)$.

The case treated by Atakishiyev and Suslov (1985) occurs when $a=(\alpha+1)/2-\mathrm{i}\gamma$, $b=(\beta+1)/2-\mathrm{i}\gamma$, $c=(\alpha+1)/2+\mathrm{i}\gamma$, $d=(\beta+1)/2+\mathrm{i}\gamma$ and the x above is replaced by $-x/2$. The notation above was used since it fits in with the notation for more general orthogonal polynomials. See the chart at the end of Askey and Wilson (1985) for the classical type orthogonal polynomials that can be given as hypergeometric polynomials. There is one change which should be made in this chart. The symmetric continuous Hahn polynomials should be moved up a line and replaced by the continuous Hahn polynomials.

There is one name for a set of polynomials that was used by Atakishiyev and Suslov (1985) that is unfortunate. They remark that Meixner polynomials and what they call Pollaczek polynomials are really the same polynomials, but with a discrete variable in the first case and a continuous variable in the second. The polynomials they call Pollaczek polynomials were rediscovered by Pollaczek, but they had been found previously by Meixner (1934). They are called Meixner polynomials of the second kind by Chihara (1978). I call them Meixner-Pollaczek polynomials. The polynomials that should be called Pollaczek polynomials are different. They are orthogonal on a finite interval, and some of them can be used to study the Coulomb potentials, both attractive and repulsive; see Szegö (1975) for the case when the weight function is absolutely continuous, Charis and Ismail (1985) for the general case, and Broad (1985) for some physical applications. The symmetric Pollaczek polynomials are the random walk polynomials associated with linear growth, birth and death processes (see Askey and Ismail 1984). Finally, basic hypergeometric or q-series) extensions of all these polynomials exist. The first derivation of the Rogers–Ramanujan identities was via a q-extension of ultraspherical polynomials. Now that we know a bit about how to use the Rogers–Ramanujan identities in some physical problems (see Baxter 1982), one should look more closely at these beautiful polynomials. Many references are given in Askey and Wilson (1985).

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References

Askey R and Ismail M 1984 *Recurrence relations, continued fractions and orthogonal polynomials* (*Memoirs Am. Math. Soc.* **300**) (Providence, RI: Am. Math. Soc.)

Askey R and Wilson J 1985 *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials* (*Memoirs Am. Math. Soc.* **319**) (Providence RI: Am. Math. Soc.)

Atakishiyev N M and Suslov S K 1985 *J. Phys. A: Math. Gen.* **18** 1583

Bailey W N 1964 *Generalized Hypergeometric Series* (New York: Stechert-Hafner)

Baxter F J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic) Ch 14

Broad J T 1985 *Phys Rev* A **31** 1494

Charis J A and Ismail M 1985 *Preprint, On Sieved orthogonal polynomials V: Sieved Pollaczek polynomials* Arizona State University

Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach)

Meixner J 1934 *J. London Math. Soc.* **9** 6

Szegö G 1975 *Orthogonal Polynomials* (Providence: Am. Math. Soc.) appendix